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# AN ASYMPTOTIC MODEL FOR A THIN BIPHASIC POROVISCOELASTIC LAYER

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**Abstract**

A three-dimensional deformation problem for articular cartilage layer is considered in the framework of the linear biphasic poroviscoelastic theory. The solid matrix of biphasic layer is assumed to be transversely isotropic. A short-time leading-order asymptotic solution is obtained for a relatively thin biphasic poroviscoelastic layer bonded to a rigid impermeable substrate.

## 1. Introduction

Biological tissues like articular cartilage can be modeled as biphasic mixtures composed of a solid skeleton and fluid phase. Under applied external load, a biphasic tissue with linearly elastic solid phase and inviscid fluid phase demonstrates a time-dependent behavior and, in particular, creep and stress-relaxation phenomena (**1**). At that, the dissipative phenomena in a biphasic tissue is caused by the frictional drag of interstitial fluid flow through the porous-permeable solid matrix (**2**).

The linear biphasic theory was developed by Mow et al. (**1**), and was extended to take into account anisotropy of the solid phase (**3**), nonlinearity (**4**), and strain-dependent permeability (**5**). However, it was demonstrated (**6**) that the failure to account for either anisotropy or viscoelasticity of the articular cartilage matrix could result in flawed predictions of the tissue deformation under general external loading.

In the framework of the linear biphasic model (**1**), an asymptotic solution for the deformation problem for a relatively thin articular cartilage layer was obtained by Ateshian et al. (**7**), while the developed asymptotic approach was later discussed in (**8**). This asymptotic model was utilized to study the axisymmetric frictionless contact interaction between two cartilage layers attached to relatively rigid subchondral bones in (**9**, **10**, **11**). The asymptotic model (**7**) was generalized for the three-dimensional case (**12**) and used for solving the contact problem in the case of articular cartilage surfaces shaped as elliptic paraboloids in (**13**).

In the present paper, we obtain the short-time leading-order asymptotic solution of the deformation problem studied in (**7**) and generalized for the case of transverse isotropy and viscoelasticity of the solid matrix.

The paper is organized as follows. In Section 2, the main equations of the linear biphasic poroviscoelastic (BPVE) theory are outlined. The deformation problem formulation for a BPVE layer is given in Section 3. Asymptotic analysis for a relatively thin layer is performed

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in Section 4 in the short-time range. The main result of the paper, presented in Section 5, is the leading order asimptotics for the displacement of the surface points (called the local indentation).

## 2. Linear biphasic poroviscoelastic (BPVE) theory

According to the biphasic theory of Mow et al. (**1**), articular cartilage is modeled as a mixture consisting of a solid phase and a fluid phase. The continuity equation for a biphasic medium is

$$\nabla\cdot(\phi_{\rm f}\mathbf{v}^{\rm f}+\phi_{\rm s}\mathbf{v}^{\rm s})=0, \tag{1}$$

where $\phi_{\rm f}$ and $\phi_{\rm s}$ are the fluid and solid volume fractions, respectively, $\mathbf{v}^{\rm f}$ and $\mathbf{v}^{\rm s}$ are the corresponding solid and fluid velocities, $\nabla$ is the gradient operator. Note that the volume fractions $\phi_{\rm f}$ and $\phi_{\rm s}$ are assumed to be constant in space and time.

Under quasi-static conditions in the absence of body forces, the momentum equations for each phase are

$$\nabla\cdot\boldsymbol{\sigma}^{\rm s}-\boldsymbol{\pi}^{\rm f}=0,\quad \nabla\cdot\boldsymbol{\sigma}^{\rm f}+\boldsymbol{\pi}^{\rm f}=0. \tag{2}$$

Here, $\boldsymbol{\pi}^{\rm f}$ is the momentum exchange between the phases due to frictional drag of relative fluid flow through the porous-permeable solid matrix. Denoting by $\mathbf{k}$ the permeability tensor, we will have the following relation (**14**, **15**):

$$\boldsymbol{\pi}^{\rm f}=-\phi_{\rm f}^2\mathbf{k}^{-1}\cdot(\mathbf{v}^{\rm f}-\mathbf{v}^{\rm s}). \tag{3}$$

We note that $\phi_{\rm f}^2=(1+\alpha)^2$, where the coefficient $\alpha$ is defined as the solid content (**16**), which is given by the ratio of solidity $\phi_{\rm s}=V_{\rm s}/V_{\rm t}$ to porosity $\phi_{\rm f}=V_{\rm f}/V_{\rm t}$ of the tissue. Here, $V_{\rm s}$, $V_{\rm f}$ and $V_{\rm t}$ are the solid, fluid and total volume of the tissue, respectively.

For a transversely isotropic skeleton, if $x_3=0$ is the plane of isotropy, the matrix of the permeability tensor takes the form

$$\mathbf{k}=\mathrm{diag}\,\{k_1,k_1,k_3\}.$$

Under the assumption that the interstitial fluid is inviscid, the stress tensor of the fluid phase is given by

$$\boldsymbol{\sigma}^{\rm f}=-\phi_{\rm f}p\mathbf{I}. \tag{4}$$

Here, $p$ is the true pressure of the fluid, $\mathbf{I}$ is the identity tensor.

Following (**2**, **17**), the stress-strain relation for the solid matrix is assumed in the form

$$\boldsymbol{\sigma}^{\rm s}=-\phi_{\rm s}p\mathbf{I}+\boldsymbol{\sigma}^{\rm VE}. \tag{5}$$

In the case of transversely isotropic skeleton, we have

$$\begin{aligned}
\sigma_{11}^{\rm VE}&=B_{11}^{\rm s}*\varepsilon_{11}+B_{12}^{\rm s}*\varepsilon_{22}+B_{13}^{\rm s}*\varepsilon_{33},\quad \sigma_{23}^{\rm VE}=2B_{44}^{\rm s}*\varepsilon_{23},\\
\sigma_{22}^{\rm VE}&=B_{12}^{\rm s}*\varepsilon_{11}+B_{11}^{\rm s}*\varepsilon_{22}+B_{13}^{\rm s}*\varepsilon_{33},\quad \sigma_{13}^{\rm VE}=2B_{44}^{\rm s}*\varepsilon_{13},\\
\sigma_{33}^{\rm VE}&=B_{13}^{\rm s}*\varepsilon_{11}+B_{13}^{\rm s}*\varepsilon_{22}+B_{33}^{\rm s}*\varepsilon_{33},\quad \sigma_{12}^{\rm VE}=2B_{66}^{\rm s}*\varepsilon_{12},
\end{aligned} \tag{6}$$

where $B_{11}^{\rm s}(t)$, $B_{12}^{\rm s}(t)$, $B_{13}^{\rm s}(t)$, $B_{33}^{\rm s}(t)$, and $B_{44}^{\rm s}(t)$ are independent stress-relaxation functions of the solid phase, $B^{\mathrm{s}}_{66}(t) = \big(B^{\mathrm{s}}_{11}(t) - B^{\mathrm{s}}_{12}(t)\big)/2$, while the $*$ sign denotes the Stieltjes convolution, i.e.,

$$B^{\mathrm{s}}_{kl} * \varepsilon_{ij} = \int\limits_{-\infty}^{t} B^{\mathrm{s}}_{kl}(t-\tau)\, d\varepsilon_{ij}(\tau).$$

At that, the strain tensor is given by

$$\boldsymbol{\varepsilon} = \frac{1}{2}\big(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{T}}\big) \tag{7}$$

with $\mathbf{u}$ being the displacement vector of the solid phase such that the solid velocity is $\mathbf{v}^{\mathrm{s}} = \partial\mathbf{u}/\partial t$, where $t$ is a time variable.

The total stress in the biphasic material, which is defined as the sum

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{s}} + \boldsymbol{\sigma}^{\mathrm{f}},$$

in light of (4) and (5) is given by

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\sigma}^{\mathrm{VE}}, \tag{8}$$

while from (2) it follows that

$$\nabla \cdot \boldsymbol{\sigma} = \mathbf{0}. \tag{9}$$

We note that the constitutive relationship (8) accounts for the inherent viscoelasticity of the solid matrix.

Finally, in light of the relation $\phi_{\mathrm{f}} + \phi_{\mathrm{s}} = 1$, the continuity equation (1) can be recast as follows:

$$\nabla \cdot \big(\mathbf{v}^{\mathrm{s}} + \phi_{\mathrm{f}}(\mathbf{v}^{\mathrm{f}} - \mathbf{v}^{\mathrm{s}})\big) = 0. \tag{10}$$

On the other hand, the second momentum equation (2), in view of (3) and (4), will take the form

$$\nabla p + \phi_{\mathrm{f}}\mathbf{k}^{-1} \cdot (\mathbf{v}^{\mathrm{f}} - \mathbf{v}^{\mathrm{s}}) = 0. \tag{11}$$

Therefore, from (10) and (11), it follows that

$$\frac{\partial}{\partial t}\nabla \cdot \mathbf{u} = \nabla \cdot (\mathbf{k} \cdot \nabla p), \tag{12}$$

where $\nabla \cdot \mathbf{u}$ is the dilatation of the solid matrix.

## 3. Deformation problem formulation for a BPVE layer

Let us consider a thin transversely isotropic biphasic layer of uniform thickness, $h$, ideally bonded to a rigid impermeable substrate and loaded by a normal load, $q$, variable-in-time (see Fig. 1). In what follows, the two-dimensional Cartesian coordinate system $(x_1, x_2)$ in the plane of the biphasic layer will be denoted as $\mathbf{y} = (y_1, y_2)$, so that $\mathbf{x} = (\mathbf{y}, z)$, where $z$ is the normal coordinate. Also, the displacement vector of the solid matrix is represented as $\mathbf{u} = (\mathbf{v}, w)$, where $\mathbf{v}$ and $w$ are the in-plane displacement vector and the normal displacement, respectively. (Do not confuse $\mathbf{v}$, the in-plane displacement, with the velocities $\mathbf{v}^{\mathrm{f}}$ and $\mathbf{v}^{\mathrm{s}}$.)

According to Eqs. (7)–(9) and (12), the equilibrium equations of the solid matrix take the form

$$B^{\rm s}_{66}*\Delta_y\mathbf{v}+(B^{\rm s}_{11}-B^{\rm s}_{66})*\nabla_y\nabla_y\cdot\mathbf{v}+B^{\rm s}_{44}*\frac{\partial^2\mathbf{v}}{\partial z^2} + (B^{\rm s}_{13}+B^{\rm s}_{44})*\frac{\partial}{\partial z}\nabla_y w = \nabla_y p, \tag{13}$$

$$B^{\rm s}_{44}*\Delta_y w+B^{\rm s}_{33}*\frac{\partial^2 w}{\partial z^2}+(B^{\rm s}_{13}+B^{\rm s}_{44})*\frac{\partial}{\partial z}\nabla_y\cdot\mathbf{v} = \frac{\partial p}{\partial z}. \tag{14}$$

Here, $\nabla_y=(\partial/\partial y_1)\mathbf{e}_1+(\partial/\partial y_2)\mathbf{e}_2$ and $\Delta_y=\nabla_y\cdot\nabla_y$ are the in-plane Hamilton and Laplace operators, respectively, while the scalar product is denoted by a dot.

In view of (1) and (3), the continuity equation for the BPVE medium has the same form as for biphasic mixtures, i.e.,

$$\frac{\partial}{\partial t}\Bigl(\nabla_y\cdot\mathbf{v}+\frac{\partial w}{\partial z}\Bigr)=k_1\Delta_y p+k_3\frac{\partial^2 p}{\partial z^2}. \tag{15}$$

Assuming that a biphasic layer is firmly attached to a rigid impermeable substrate, on the bottom of the layer, $z=h$, we impose the boundary conditions

$$\mathbf{v}\big|_{z=h}=0,\quad w\big|_{z=h}=0,\quad \frac{\partial p}{\partial z}\bigg|_{z=h}=0. \tag{16}$$

On the upper surface, $z=0$, the layer is assumed to be loaded only by a variable distributed normal load $q$, so that the traction boundary conditions

$$\sigma_{33}\big|_{z=0}=-q,\quad \sigma_{13}\big|_{z=0}=\sigma_{23}\big|_{z=0}=0$$

can be rewritten as follows (see Eqs. (5) and (6)):

$$-p+B^{\rm s}_{13}*\nabla_y\cdot\mathbf{v}+B^{\rm s}_{33}*\frac{\partial w}{\partial z}\bigg|_{z=0}=-q, \tag{17}$$

$$B^{\rm s}_{44}*\Bigl(\nabla_y w+\frac{\partial\mathbf{v}}{\partial z}\Bigr)\bigg|_{z=0}=0. \tag{18}$$

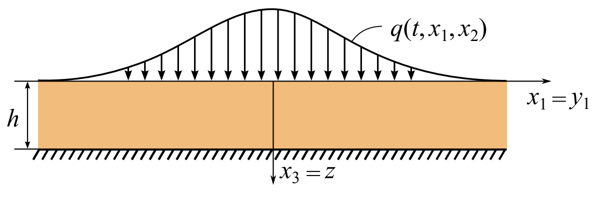


**Figure 1** A biphasic poroviscoelastic layer bonded to a rigid impermeable substrate and supporting a time-dependent normal load

Moreover, assuming that the normal load $q$ is transferred from an impermeable punch, we require that

$$\frac{\partial p}{\partial z}\Big|_{z=0} = 0, \tag{19}$$

that is no fluid flow takes place at the contact interface.

Equations (13)–(15) with the boundary conditions (16)–(19) and the zero initial conditions

$$\mathbf{v} = 0, \quad w = 0, \quad p = 0, \quad -\infty < t < 0, \tag{20}$$

constitute the deformation problem for a thin BPVE layer.

## 4. Short-time asymptotic analysis of the deformation problem

Introducing a characteristic length, $h_*$, and a small positive parameter, $\varepsilon$, we require that

$$h = \varepsilon h_*.$$

Moreover, as usual, we introduce the dimensionless in-plane coordinates

$$\boldsymbol{\eta} = (\eta_1, \eta_2), \quad \eta_i = \frac{y_i}{h_*}, \quad i = 1, 2,$$

and stretch the normal coordinate

$$\zeta = \varepsilon^{-1}\frac{z}{h_*}.$$

The governing equations will be non-dimensionalized using the following non-dimensional variables:

$$\tau = \frac{k_3 B_{44}^{\mathrm{s}0}}{h^2}t, \quad \mathbf{V} = \frac{\mathbf{v}}{h}, \quad W = \frac{w}{h}, \quad P = \frac{p}{B_{44}^{\mathrm{s}0}}, \quad Q = \frac{q}{B_{44}^{\mathrm{s}0}}. \tag{21}$$

Here, $B_{44}^{\mathrm{s}0} = B_{44}^{\mathrm{s}}(0)$ is the instantaneous modulus.

After that, we apply to the obtained system the Laplace transformation in dimensional time, i.e.,

$$\tilde{P}(\rho) = \int\limits_0^\infty P\Big(\frac{h^2}{k_3 B_{44}^{\mathrm{s}0}}\tau\Big)\exp(-\rho\tau)\,d\tau,$$

and arrive at the following problem:

$$\bar{b}_{44}^{\mathrm{s}}\frac{\partial^2\tilde{\mathbf{V}}}{\partial\zeta^2} + \varepsilon(\bar{b}_{13}^{\mathrm{s}} + \bar{b}_{44}^{\mathrm{s}})\nabla_\eta\frac{\partial\tilde{W}}{\partial\zeta} + \varepsilon^2\big(\bar{b}_{66}^{\mathrm{s}}\Delta_\eta\tilde{\mathbf{V}} + (\bar{b}_{11}^{\mathrm{s}} - \bar{b}_{66}^{\mathrm{s}})\nabla_\eta\nabla_\eta\cdot\tilde{\mathbf{V}}\big) = \varepsilon\nabla_\eta\tilde{P}, \tag{22}$$

$$\bar{b}_{33}^{\mathrm{s}}\frac{\partial^2\tilde{W}}{\partial\zeta^2} + \varepsilon(\bar{b}_{13}^{\mathrm{s}} + \bar{b}_{44}^{\mathrm{s}})\nabla_\eta\cdot\frac{\partial\tilde{\mathbf{V}}}{\partial\zeta} + \varepsilon^2\bar{b}_{44}^{\mathrm{s}}\Delta_\eta\tilde{W} = \frac{\partial\tilde{P}}{\partial\zeta}, \tag{23}$$

$$\rho\Big(\frac{\partial\tilde{W}}{\partial\zeta} + \varepsilon\nabla_\eta\cdot\tilde{\mathbf{V}}\Big) = \frac{\partial^2\tilde{P}}{\partial\zeta^2} + \varepsilon^2\kappa_1\Delta_\eta\tilde{P}, \tag{24}$$

$$\tilde{\mathbf{V}}\big|_{\zeta=1} = 0, \quad \tilde{W}\big|_{\zeta=1} = 0, \quad \frac{\partial\tilde{P}}{\partial\zeta}\Big|_{\zeta=1} = 0, \tag{25}$$

$$-\tilde{P} + \bar{b}_{13}^{\rm s} \nabla_\eta \cdot \tilde{\mathbf{V}} + \bar{b}_{33}^{\rm s} \frac{\partial \tilde{W}}{\partial \zeta}\bigg|_{\zeta=0} = -\tilde{Q}, \tag{26}$$

$$\nabla_\eta \tilde{W} + \frac{\partial \tilde{\mathbf{V}}}{\partial \zeta}\bigg|_{\zeta=0} = 0, \quad \frac{\partial \tilde{P}}{\partial \zeta}\bigg|_{\zeta=0} = 0. \tag{27}$$

Here the Laplace transforms are denoted by a tilde, $\rho$ is the Laplace transformation parameter, $\kappa_1 = k_1/k_3$, and $\bar{b}_{kl}^{\rm s} = \rho \tilde{B}_{kl}^{\rm s}/B_{44}^{\rm s0}$, where $\tilde{B}_{kl}^{\rm s}$ is the Laplace transform of $B_{kl}^{\rm s}\big(h^2\tau/(B_{44}^{\rm s0}k_3)\big)$ with respect to the dimensionless time variable $\tau$.

Following Ateshian et al. (**7**), we represent the asymptotic ansatz for the solution to the system (22)–(27) in the form

$$\tilde{P} \simeq \tilde{Q} + \varepsilon^2 \tilde{P}^1, \quad \tilde{\mathbf{V}} \simeq \varepsilon \tilde{\mathbf{V}}^0, \quad \tilde{W} \simeq \varepsilon^2 \tilde{W}^0. \tag{28}$$

Substituting the expressions into Eqs. (22)–(24) and the boundary conditions (25)–(27), after some simple calculations, we get

$$\tilde{\mathbf{V}}^0 = -\frac{1}{2\bar{b}_{44}^{\rm s}}(1-\zeta^2)\nabla_\eta \tilde{Q}, \tag{29}$$

while the pair $\tilde{W}^0$ and $\tilde{P}^1$ should be determined as the solution of the problem

$$\begin{aligned} \bar{b}_{33}^{\rm s}\frac{\partial^2 \tilde{W}^0}{\partial \zeta^2} - \frac{\partial \tilde{P}^1}{\partial \zeta} &= -(\bar{b}_{44}^{\rm s} + \bar{b}_{13}^{\rm s})\nabla_\eta \cdot \frac{\partial \tilde{\mathbf{V}}^0}{\partial \zeta}, \\ \rho\frac{\partial \tilde{W}^0}{\partial \zeta} - \frac{\partial^2 \tilde{P}^1}{\partial \zeta^2} &= \kappa_1 \Delta_\eta \tilde{Q} - \rho \nabla_\eta \cdot \tilde{\mathbf{V}}^0, \end{aligned} \tag{30}$$

$$\begin{aligned} &\tilde{W}^0\big|_{\zeta=1} = 0, \quad \frac{\partial \tilde{P}^1}{\partial \zeta}\bigg|_{\zeta=1} = 0, \\ -\tilde{P}^1 + \bar{b}_{33}^{\rm s}&\frac{\partial \tilde{W}^0}{\partial \zeta}\bigg|_{\zeta=0} = -\bar{b}_{13}^{\rm s}\nabla_\eta \cdot \tilde{\mathbf{V}}^0\big|_{\zeta=0}, \quad \frac{\partial \tilde{P}^1}{\partial \zeta}\bigg|_{\zeta=0} = 0. \end{aligned} \tag{31}$$

The general solution of the homogeneous differential system corresponding to (30) is given by

$$\begin{aligned} \tilde{W}_0^0 &= C_0 + C_1 \cosh\sqrt{f(\rho)}\zeta + C_2 \sinh\sqrt{f(\rho)}\zeta, \\ \tilde{P}_0^1 &= C_3 + \frac{\rho}{\sqrt{f(\rho)}}\big(C_1 \sinh\sqrt{f(\rho)}\zeta + C_2 \cosh\sqrt{f(\rho)}\zeta\big), \end{aligned} \tag{32}$$

where we introduced the notation

$$f(\rho) = \frac{\rho}{\bar{b}_{33}^{\rm s}}.$$

It can be shown that, in view of (29), the following pair represents a particular solution of the system (30):

$$\begin{aligned} \tilde{W}_1^0 &= \left(\frac{2}{\rho}\left[\bar{b}_{44}^{\rm s} + \bar{b}_{13}^{\rm s} - \bar{b}_{33}^{\rm s} + \kappa_1 \bar{b}_{44}^{\rm s}\right] + 1 - \frac{\zeta^2}{6}\right)\frac{\zeta}{2\bar{b}_{44}^{\rm s}}\Delta_\eta \tilde{Q}, \\ \tilde{P}_1^1 &= \frac{(\bar{b}_{44}^{\rm s} + \bar{b}_{13}^{\rm s} - \bar{b}_{33}^{\rm s})}{2\bar{b}_{44}^{\rm s}}\zeta^2 \Delta_\eta \tilde{Q}. \end{aligned} \tag{33}$$

Now, substituting the expressions

$$\tilde{W}^0 = \tilde{W}_0^0 + \tilde{W}_1^0, \quad \tilde{P}^1 = \tilde{P}_0^1 + \tilde{P}_1^1 \tag{34}$$

into the system of boundary conditions (31) and taking into account Eq. (29), we evaluate the integration constants $C_0$, $C_1$, $C_2$, and $C_3$ as follows:

$$C_0 = -\Big(\frac{1}{3\bar{b}_{44}^{\rm s}} + \frac{\kappa_1}{\rho}\Big)\Delta_\eta \tilde{Q}, \quad C_1 = 0, \quad C_2 = -\frac{(\bar{b}_{44}^{\rm s} + \bar{b}_{13}^{\rm s} - \bar{b}_{33}^{\rm s})}{\rho \bar{b}_{44}^{\rm s}} \frac{\Delta_\eta \tilde{Q}}{\sinh\sqrt{f(s)}},$$

$$C_3 = \frac{\bar{b}_{33}^{\rm s}}{2\bar{b}_{44}^{\rm s}}\left(\frac{2}{\rho}(\bar{b}_{44}^{\rm s} + \bar{b}_{13}^{\rm s} - \bar{b}_{33}^{\rm s} + \kappa_1 \bar{b}_{44}^{\rm s}) + 1 - \frac{\bar{b}_{13}^{\rm s}}{\bar{b}_{33}^{\rm s}}\right)\Delta_\eta \tilde{Q}. \tag{35}$$

Thus, the functions $W$, $\mathbf{V}$, and $P$ now can be obtained by performing the inverse Laplace transform.

## 5. Local indentation of a thin BPVE layer

Recall that $B_{44}^{\rm s}(t)$ represents the out-of-plane relaxation modulus in shear, so that, in view of the initial conditions (20), Eqs. $(6)_2$ and $(6)_4$ take the form

$$\sigma_{3i}(t) = 2\int_{0^-}^{t} B_{44}^{\rm s}(t-\tau)\dot{\varepsilon}_{3i}(\tau)\, d\tau, \quad i = 1, 2. \tag{36}$$

Note that in view of (5) and (8) we have $\sigma_{3i} = \sigma_{3i}^{\rm s} = \sigma_{3i}^{\rm VE}$ for $i = 1, 2$.

Let us introduce the out-of-plane creep compliance in shear of the solid matrix, $J_{44}^{\rm s}(t)$, which governs the deformation response of the solid phase under application of a step out-of-plane shear stress of unit magnitude. Hence, the inverse relations for (36) are given by

$$2\varepsilon_{3i}(t) = \int_{0^-}^{t} J_{44}^{\rm s}(t-\tau)\dot{\sigma}_{3i}(\tau)\, d\tau, \quad i = 1, 2.$$

For a given relaxation modulus $B_{44}^{\rm s}(t)$ and its Laplace transform $\tilde{B}_{44}^{\rm s}(s)$ (with respect to the time variable $t$), the corresponding creep compliance can be evaluated through its Laplace transform

$$\tilde{J}_{44}^{\rm s}(s) = \frac{1}{s^2 \tilde{B}_{44}^{\rm s}(s)}. \tag{37}$$

We observe that in light of (32)–(35) we will have

$$\tilde{W}^0\big|_{\zeta=0} = -\Big(\frac{1}{3\bar{b}_{44}^{\rm s}} + \frac{\kappa_1}{\rho}\Big)\Delta_\eta \tilde{Q},$$

where $\bar{b}_{44}^{\rm s} = \rho \tilde{B}_{44}^{\rm s}/B_{44}^{\rm s0}$.

Thus, collecting formulas (21), (28), (29), (32)–(35), and taking account of (37), we obtain the following asymptotic representations for the displacements of the surface points of the bonded thin BPVE layer:

$$\mathbf{v}\big|_{z=0} \simeq -\frac{h^2}{2}\int\limits_{0^-}^{t} J^{\mathrm{s}}_{44}(t-\tau)\frac{\partial}{\partial\tau}\nabla_y q(\tau,\mathbf{y})\,d\tau, \tag{38}$$

$$w\big|_{z=0} \simeq -\frac{h^3}{3}\int\limits_{0^-}^{t} J^{\mathrm{s}}_{44}(t-\tau)\frac{\partial}{\partial\tau}\Delta_y q(\tau,\mathbf{y})\,d\tau - hk_1\int\limits_{0}^{t}\Delta_y q(\tau,\mathbf{y})\,d\tau. \tag{39}$$

It should be emphasized that the approximate solution obtained above represents the short-time asymptotics. Indeed, the first formula (21) can be rewritten as $\tau = \varepsilon^{-2}(k_3 B^{\mathrm{s}0}_{44}/h_*^2)t$, so that a finite interval for the fast variable $\tau$ corresponds to a very short interval for the original time variable $t$.

Observe that formula (39) reflects two types of mechanisms, which are responsible for time-dependent effects in articular cartilage: flow independent and flow dependent characterized by the first and second terms on the right-hand side of (39), respectively.

The asymptotic relations (38) and (39) can be used to formulate asymptotic models (**18**) for the frictionless contact interaction between bonded thin BPVE layers. With regard to the case of the frictionless contact interaction between articular cartilage layers, the developed asymptotic solution allows us to account for the inherent viscoelasticity of the solid matrix.

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